Neural Computing and Applications (2020) 32:10847–10857
https://doi.org/10.1007/s00521-018-3683-y

S.I. : INDIA INTL. CONGRESS ON COMPUTATIONAL INTELLIGENCE 2017

Portfolio optimization in fuzzy asset management with coherent risk measures derived from risk averse utility

Yuji Yoshida[1]

Received: 22 March 2018 / Accepted: 9 August 2018 / Published online: 17 August 2018

Abstract

A portfolio optimization problem with fuzzy random variables is discussed using coherent risk measures, which are characterized by weighted average value-at-risks with risk spectra. By perception-based approach, coherent risk measures and weighted average value-at-risks are extended for fuzzy random variables. Coherent risk measures derived from risk averse utility functions are introduced to discuss the portfolio optimization with randomness and fuzziness. The randomness is estimated by probability, and the fuzziness is evaluated by lambda-mean functions and evaluation weights. By mathematical programming approaches, a solution is derived for the risk-minimizing portfolio optimization problem. Numerical examples are given to compare coherent risk measures. It is made clear that coherent risk measures derived from risk averse utility functions have excellent properties as risk criteria for these optimization problems. Not only pessimistic and necessity case but also optimistic and possibility case are calculated numerically to deal with uncertain information.

Keywords Coherent risk measure · Fuzzy random variable · Perception-based extension · Weighted average value-at-risk · Portfolio optimization

List of symbols

VaR, AVaR	Value-at-risk and average value-at-risk
AVaR^v ($\widetilde{\mathrm{AVaR}}^v$)	(Extended) Weighted average value-at-risk with v
ρ ($\tilde{\rho}$)	(Extended) Coherent risk measure
v, C	Risk spectrum and its component function
$\mathcal{N}$	The set of all fuzzy numbers
$\tilde{n}, \tilde{n}_\alpha = [\tilde{n}_\alpha^-, \tilde{n}_\alpha^+]$	Fuzzy number and its α-cut
$\tilde{X}, \tilde{X}_\alpha = [\tilde{X}_\alpha^-, \tilde{X}_\alpha^+]$	Fuzzy random variable and its α-cut
$\mathcal{X}$ ($\tilde{\mathcal{X}}$)	The family of all integrable real-valued (fuzzy-valued) random variables
$E, \tilde{E}$	Expectation and perception-based expectation
E^λ	Mean of fuzzy numbers
λ	Optimistic/pessimistic index
$w(\alpha)$	Possibility/necessity evaluation weight
f	Utility function
S_t^i ($\tilde{S}_t^i$)	(Fuzzy-valued) Stock price for asset i at time t
R_t^i ($\tilde{R}_t^i$)	(Fuzzy-valued) Rate of return for asset i at time t
$w_t = (w_t^1, \ldots, w_t^n)$	Portfolio weight vector
$\mathcal{W}_t$	The set of all portfolio weight vectors
$\mu_t = [\mu_t^i]$	Vector of expected rates of return
$\Sigma_t = [\sigma_t^{ij}]$	Variance–covariance matrix for rates of return
γ_t^*	The optimal expected rate of return
ρ_t^*	The optimal risk value

1 Introduction

In financial asset management, portfolio allocation is a technique to achieve both minimization of asset risks and maximization of expected returns. In classical mean-variance portfolio models, the variance is used as a risk measure [19]. Recently drastic declines of asset prices are

✉ Yuji Yoshida
yoshida@kitakyu-u.ac.jp

[1] Faculty of Economics and Business Administration, The University of Kitakyushu, 4-2-1 Kitagata, Kokuraminami, Kitakyushu 802-8577, Japan

studied, and *value-at-risk* is used widely to estimate the risk of asset price declines in practical financial management [12]. Value-at-risk is defined by percentiles at a specified probability; however, it does not have coherency. *Coherent risk measures* have been studied to improve the criterion of risks with worst scenarios [4]. Several improved risk measures based on value-at-risks are proposed: for example, conditional value-at-risk, expected shortfall and entropic value-at-risk [23, 29]. Kusuoka [16] gave a spectral representation for coherent risk measures, and Acerbi [1] and Adam et al. [2] discussed its applications to portfolio selection and so on. Emmer at al. [7] compared risk measures by their properties to find best risk measures. This paper deals with portfolio optimization using weighted average value-at-risk representation of coherent risk measures in fuzzy environment.

Portfolio optimization in fuzzy logic framework was studied first from *decision making with fuzzy goal*, which was introduced by Bellman and Zadeh [5], and it has been developed with possibility measures and necessity measures by Tanaka and Guo [27], Tanaka et al. [28], Watada [31], Katagiri at al. [13] and so on. These surveys can be found in Fang et al. [8]. Using fuzzy random variables, *maximization of fuzzy variable returns* is studied by Hasuike et al. [11], Li and Xu [18], Sadati and Doniavi [25], Sadati and Nematian [26] and so on. On the other hand, Yoshida [35], Wang et al. [30] and Moussa et al. [20] have studied risk values using *value-at-risks for fuzzy random variables* as risk criteria, and further Yoshida [39] has discussed portfolio optimization with *coherent risk measures* in fuzzy environment.

During the financial crisis in September 2008 and the China's stock market crashing in May 2015, we have experienced the serious distrust about the stock market because of imprecise information among investors and the market. Fuzzy logic is an important tool to represent this kind of linguistic uncertainty [10, 14]. In this paper, to represent uncertainty we use *fuzzy random variables* which have two kinds of uncertainties, i.e., randomness and fuzziness. Fuzzy random variables are applied to decision making under uncertainty with fuzziness such as linguistic information in engineering, economics et al. [17]. Risk measures for real-valued random variables are extended for fuzzy random variables by perception-based approach in [33]. Yoshda [32] introduced the mean and the variance of fuzzy random variables, using *λ-mean functions* and *evaluation weights*. This paper estimates fuzzy random variables by probabilistic expectation and these criteria, which are characterized by decision maker's *pessimistic–optimistic indexes* and *possibility–necessity weights*. These parameters are decided by the investor with his certainty about information in the stock market.

On February 5, 2018, the flash crash of the stock market has occurred because of high-speed computers trading. Nowadays institutional investors operate high-speed computers based on neural computing and deep learning, and the high-speed trading among computers causes the flash crash. Decision makers usually select trading strategies after measuring and observing the risk of assets in the market. For quick and stable trading, *we need to take risk criterion based on investor's utility into computational decision making* in asset management. Yoshida [38] has dealt with portfolio optimization with coherent risk measures; however, it could not demonstrate how we select proper coherent risk measures for utility functions. Recently Yoshida [39] has studied the mathematical relation between decision maker's risk averse utility functions and coherent risk measures, and it has derived *coherent risk measures adapted to decision maker's risk averse utility*. The derived coherent risk measure can inherit the risk averse property of the decision maker's utility function as risk spectrum weighting. On the basis of mathematical results in [39], this paper introduces coherent risk measures derived from utility functions and we discuss risk-minimizing portfolio optimization with fuzzy random variables. We give numerical examples to compare coherent risk measures, and we demonstrate this optimization method which brings us reasonable and stable results taking over decision maker's risk averse utility.

The paper is organized as follows: In Sect. 2, we introduce coherent risk measures and their spectral representation. In Sect. 3, from [39] we introduce *weighted average value-at-risks* as coherent risk measures derived from decision maker's utility functions. In Sect. 4, we give fuzzy numbers and fuzzy random variables, and we define extended estimations for fuzzy random variables by perception-based approach. In Sect. 5, we introduce scalarization tools with λ-mean functions and evaluation weights in order to evaluate the randomness and fuzziness for fuzzy random variables. In Sect. 6, using coherent risk measures and weighted average value-at-risks, we discuss portfolio optimization under uncertainty in three steps: The first step is mean-variance portfolio optimization, the second step is risk-sensitive portfolio optimization, and in the last step, we obtain a solution of portfolio optimization for risk minimization. In Sect. 7, we investigate numerical examples for the obtained results and we compare coherent risk measures in relation to utility functions from the numerical results.

2 Coherent risk measures

Let Ω be a sample space and let P be a non-atomic probability measure on Ω. Let $\mathcal{X}$ be a family of all integrable real-valued random variables X on Ω for which there exists a non-empty open interval I such that their cumulative distribution functions $F_X(\cdot) = P(X < \cdot) : I \to (0,1)$ are continuous, strictly increasing and onto. Then, there exist strictly increasing and continuous inverse functions F_X^{-1}. For a positive probability p, a *value-at-risk* is defined by the following percentile of the distribution function:

$$\mathrm{VaR}_p(X) = \sup\{x \in I \mid F_X(x) \le p\} = F_X^{-1}(p) \qquad (1)$$

for $p \in (0,1)$ and $\mathrm{VaR}_1(X) = \sup I$. Then, an *average value-at-risk* is given by

$$\mathrm{AVaR}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_q(X)\,\mathrm{d}q \qquad (2)$$

for $p \in (0,1]$ and $X \in \mathcal{X}$. Let $\mathbf{R} = (-\infty, \infty)$ and $\mathbf{R}_+ = [0, \infty)$. The following definitions are introduced to characterize risk measures.

Definition 1 Let a map $\rho : \mathcal{X} \mapsto \mathbf{R}$.

- (i) Two random variables $X(\in \mathcal{X})$ and $Y(\in \mathcal{X})$ are called *comonotonic* if $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \ge 0$ for almost all $\omega, \omega' \in \Omega$.
- (ii) ρ is called *comonotonically additive* if $\rho(X + Y) = \rho(X) + \rho(Y)$ for all comonotonic random variables $X, Y \in \mathcal{X}$.
- (iii) ρ is called *law invariant* if $\rho(X) = \rho(Y)$ for all $X, Y \in \mathcal{X}$ satisfying $P(X < \cdot) = P(Y < \cdot)$.
- (iv) ρ is called *continuous* if $\lim_{n\to\infty} \rho(X_n) = \rho(X)$ for $\{X_n\} \subset \mathcal{X}$ and $X \in \mathcal{X}$ such that $\lim_{n\to\infty} X_n = X$ almost surely.

Definition 2 [4] A map $\rho : \mathcal{X} \mapsto \mathbf{R}$ is called a *coherent risk measure* if it satisfies the following (i)–(iv):

- (i) $\rho(X) \ge \rho(Y)$ for $X, Y \in \mathcal{X}$ satisfying $X \le Y$. (*monotonicity*)
- (ii) $\rho(X + c) = \rho(X) - c$ for $X \in \mathcal{X}$ and $c \in \mathbf{R}$. (*translation invariance*)
- (iii) $\rho(cX) = c\,\rho(X)$ for $X \in \mathcal{X}$ and $c \in \mathbf{R}_+$. (*positive homogeneity*)
- (iv) $\rho(X + Y) \le \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{X}$. (*subadditivity*)

In [4, 37], it is known about value-at-risks (1) and average value-at-risks (2) that $-\mathrm{AVaR}_p(\cdot)$ *is a coherent risk measure however* $-\mathrm{VaR}_p(\cdot)$ *is not coherent*. For a probability $p \in (0,1]$ and a function v on [0, 1], we define an average value-at-risk with weighting v on (0, p) by

$$\mathrm{AVaR}_p^v(X) = \int_0^p \mathrm{VaR}_q(X)\,v(q)\,\mathrm{d}q \Big/ \int_0^p v(q)\,\mathrm{d}q. \qquad (3)$$

Hence, (3) is called a *weighted average value-at-risk* and v is called a *risk spectrum* if it is non-increasing. Then, coherent risk measures have the following *spectral representation* [16, 39].

Lemma 1 *Let $\rho : \mathcal{X} \mapsto \mathbf{R}$ be a law invariant, comonotonically additive, continuous coherent risk measure. Then, there exists a risk spectrum v such that*

$$\rho(X) = -\int_0^1 \mathrm{VaR}_q(X)\,v(q)\,\mathrm{d}q = -\mathrm{AVaR}_1^v(X) \qquad (4)$$

for $X \in \mathcal{X}$. Further, $-\mathrm{AVaR}_p^v$ is a coherent risk measure on $\mathcal{X}$ for $p \in (0,1)$.

Because $-\mathrm{AVaR}_p^v$ is a coherent risk measure in Lemma 1, the following lemma holds from Definition 2.

Lemma 2 *Let $X, Y \in \mathcal{X}$. Then, the following (i)–(iv) hold:*

- (i) *If $X \le Y$, then $\mathrm{AVaR}_p^v(X) \le \mathrm{AVaR}_p^v(Y)$.*
- (ii) $\mathrm{AVaR}_p^v(X + c) = \mathrm{AVaR}_p^v(X) + c$ *for* $c \in \mathbf{R}$.
- (iii) $\mathrm{AVaR}_p^v(cX) = c\,\mathrm{AVaR}_p^v(X)$ *for* $c \in \mathbf{R}_+$.
- (iv) $\mathrm{AVaR}_p^v(X + Y) \ge \mathrm{AVaR}_p^v(X) + \mathrm{AVaR}_p^v(Y)$.

3 Coherent risk measures derived from decision maker's utility

Yoshida [39] has studied coherent risk measures adapted to decision maker's risk averse utility, using weighted average value-at-risks. Let a *risk averse exponential utility function*

$$f(x) = \frac{1 - e^{-\tau x}}{\tau} \qquad (5)$$

for $x \in \mathbf{R}$ with a positive constant τ. On the basis of mathematical results in [39], we introduce coherent risk measures derived from risk averse utility functions f. Let a probability $p \in (0,1]$. Under decision maker's utility function f, the average value-at-risks of random variables $X(\in \mathcal{X})$ over (0, p) are estimated as

$$f^{-1}\left(\frac{1}{p}\int_0^p f(\mathrm{VaR}_q(X))\,\mathrm{d}q\right). \qquad (6)$$

In portfolio optimization, we use a coherent risk measure $-\mathrm{AVaR}_p^v$ given by (3) which is the nearest to the risk estimation (6). Let v be a risk spectrum attaining the minimum distance

$$\min_v \sum_{X\in\mathcal{X}} \left(f^{-1}\left(\frac{1}{p}\int_0^p f(\mathrm{VaR}_q(X))\,\mathrm{d}q\right) - \mathrm{AVaR}_p^v(X)\right)^2 \qquad (7)$$

for $p \in (0,1]$. Then, the risk estimation (6) on the downside range $(-\infty, \mathrm{VaR}_p(X))$ is related to negative utilities, and it acquires decision maker's risky sense regarding random variable X, while the coherent risk measure $-\mathrm{AVaR}_p^{\nu}$ is given by AVaR_p^{ν} which has a kind of semi-linearity such as Lemma 2(ii), (iii). In this paper, we deal with a case when value-at-risks of random variables $X \in \mathcal{X}$ are represented as

$$\mathrm{VaR}_p(X) = \mu + \kappa(p) \cdot \sigma \tag{8}$$

with a mean $\mu = E(X)$ and a standard deviation $\sigma = \sigma(X)$, where $\kappa : (0,1) \mapsto \mathbf{R}$ is an increasing function. Then, we have the following lemma from [39].

Lemma 3 *Let $\nu \in \mathcal{N}$ be a function given by*

$$\nu(p) = e^{-\int_p^1 C(q)\,\mathrm{d}q} C(p) \tag{9}$$

for $p \in (0,1]$ with its component function

$$C(p) = \frac{\sum_{X\in\mathcal{X}} \sigma(X) \dfrac{f(\mathrm{VaR}_p(X)) - \frac{1}{p}\int_0^p f(\mathrm{VaR}_q(X))\,\mathrm{d}q}{pf'\left(f^{-1}\left(\frac{1}{p}\int_0^p f(\mathrm{VaR}_q(X))\,\mathrm{d}q\right)\right)}}{\sum_{X\in\mathcal{X}} \sigma(X)\left(\mathrm{VaR}_p(X) - f^{-1}\left(\frac{1}{p}\int_0^p f(\mathrm{VaR}_q(X))\,\mathrm{d}q\right)\right)}. \tag{10}$$

If ν is non-increasing, then ν is an optimal risk premium for (7).

Sketch proof Let $p \in (0,1)$. From (8), for $X \in \mathcal{X}$, we put $\mathrm{VaR}_p(X) = \mu + \kappa(p) \cdot \sigma$ with a mean $\mu = E(X)$ and a standard deviation $\sigma = \sigma(X)$. To discuss the minimization (7), by (3) we define

$$G(\underline{\nu}) = \sum_{X\in\mathcal{X}} \left(f^{-1}\left(\frac{1}{p}\int_0^p f(\mu + \kappa(q)\sigma)\,\mathrm{d}q\right) - \frac{\int_0^p (\mu + \kappa(q)\sigma)\,\underline{\nu}(q)\,\mathrm{d}q}{\int_0^p \underline{\nu}(q)\,\mathrm{d}q}\right)^2 \tag{11}$$

for risk spectra $\underline{\nu}$. Let ν be a risk spectrum attaining the minimum (7). Then, $(1-t)\nu + t\varepsilon$ is also a risk spectrum for $t \in (0,1)$ and risk spectra ε. Hence, we have

$$\lim_{t\downarrow 0} \frac{G((1-t)\nu + t\varepsilon) - G(\nu)}{t} = 0 \tag{12}$$

for any risk spectrum ε. This follows

$$\sum_{X\in\mathcal{X}} \sigma \left(f^{-1}\left(\frac{1}{p}\int_0^p f(\mu + \kappa(q)\sigma)\,\mathrm{d}q\right) - \frac{\int_0^p (\mu + \kappa(q)\sigma)\,\nu(q)\,\mathrm{d}q}{\int_0^p \nu(q)\,\mathrm{d}q}\right) = 0. \tag{13}$$

Therefore, we obtain

$$\sum_{X\in\mathcal{X}} \sigma \left(f^{-1}\left(\frac{1}{p}\int_0^p f(\mu + \kappa(q)\sigma)\,\mathrm{d}q\right) \int_0^p \nu(q)\,\mathrm{d}q - \int_0^p (\mu + \kappa(q)\sigma)\,\nu(q)\,\mathrm{d}q\right) = 0 \tag{14}$$

for all $p \in (0,1)$. Differentiating (14) with respect to p, we get

$$\frac{\nu(p)}{\int_0^p \nu(q)\,\mathrm{d}q} = C(p) \tag{15}$$

for all $p \in (0,1)$, where C is defined by (10). Thus, we obtain (9) from (15). □

4 Fuzzy random variables and risk measures

A *fuzzy number* is represented by its membership function $\tilde{n} : \mathbf{R} \to [0,1]$ which is normal, upper semicontinuous, fuzzy convex and has a compact support [40]. Let $\mathcal{N}$ be the set of all fuzzy numbers. For a fuzzy number $\tilde{n} \in \mathcal{N}$, its α-cuts are given by closed intervals $\tilde{n}_\alpha = \{x \in \mathbf{R} \mid \tilde{n}(x) \geq \alpha\} = [\tilde{n}_\alpha^-, \tilde{n}_\alpha^+]$ for $\alpha \in (0,1]$. Hence, the *fuzzy max order* $\succeq$ is a partial order defined on $\mathcal{N}$ as follows: For fuzzy numbers $\tilde{n}, \tilde{m} \in \mathcal{N}, \tilde{n} \succeq \tilde{m}$ means that $\tilde{n}_\alpha^\pm \geq \tilde{m}_\alpha^\pm$ for all $\alpha \in (0,1]$. An addition and a scalar multiplication for fuzzy numbers are defined as follows: For $\tilde{n}, \tilde{m} \in \mathcal{N}$ and $c \in \mathbf{R}$, the addition $\tilde{n} + \tilde{m}$ of $\tilde{n}$ and $\tilde{m}$ and the scalar multiplication $c\,\tilde{n}$ of c and $\tilde{n}$ are fuzzy numbers given by their α-cuts $(\tilde{n} + \tilde{m})_\alpha = [\tilde{n}_\alpha^- + \tilde{m}_\alpha^-, \tilde{n}_\alpha^+ + \tilde{m}_\alpha^+]$ and $(c\,\tilde{n})_\alpha = [c\,\tilde{n}_\alpha^-, c\,\tilde{n}_\alpha^+]$ if $c \geq 0$ and $(c\,\tilde{n})_\alpha = [c\,\tilde{n}_\alpha^+, c\,\tilde{n}_\alpha^-]$ if $c < 0$, where $\tilde{n}_\alpha = [\tilde{n}_\alpha^-, \tilde{n}_\alpha^+]$ and $\tilde{m}_\alpha = [\tilde{m}_\alpha^-, \tilde{m}_\alpha^+]$ $(\alpha \in [0,1])$. Now we also use the following *metric* d_H on $\mathcal{N}$ induced from Hausdorff metric: $d_H(\tilde{n}, \tilde{m}) = \sup_{\alpha\in[0,1]} \max\{|\tilde{n}_\alpha^- - \tilde{m}_\alpha^-|, |\tilde{n}_\alpha^+ - \tilde{m}_\alpha^+|\}$ for $\tilde{n}, \tilde{m} \in \mathcal{N}$. A fuzzy-number-valued map $\tilde{X} : \Omega \to \mathcal{N}$ is called a *fuzzy random variable* if $\tilde{X}_\alpha^\pm \in \mathcal{X}$ for all $\alpha \in (0,1]$, where $\tilde{X}_\alpha(\omega) = \{x \in \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)]$ for $\omega \in \Omega$. Then, a fuzzy random variable $\tilde{X}$ is called integrable if random variables $\tilde{X}_\alpha^\pm$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{\mathcal{X}}$ be the family of all integrable fuzzy random variables on Ω. Kruse and Meyer [15] gave the expectation of fuzzy random variables $\tilde{X} \in \tilde{\mathcal{X}}$ in the following *perception-based approach*, which is based on Zadeh's extension principle, as follows:

$$\tilde{E}(\tilde{X})(x) = \sup_{X\in\mathcal{X}:E(X)=x} \inf_{\omega\in\Omega} \tilde{X}(\omega)(X(\omega)) \tag{16}$$

for $x \in \mathbf{R}$, where $E(X) = \int X dP$ is the expectation for real-valued random variables $X \in \mathcal{X}$. Then, the expectation $\tilde{E}(\tilde{X})$ is a fuzzy number with α-cuts $\tilde{E}(\tilde{X})_\alpha = [E(\tilde{X}_\alpha^-), E(\tilde{X}_\alpha^+)]$. Puri and Ralescu [22] also discussed the expectation (16) of fuzzy random variables by Aumann integral. For a weighted average value-at-risk AVaR_p^ν in Sect. 3, we introduce its extended version for a fuzzy random variable $\tilde{X} \in \tilde{\mathcal{X}}$ by a fuzzy number

$$\widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X})(x) = \sup_{X \in \mathcal{X} : \mathrm{AVaR}_p^\nu(X) = x} \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)) \tag{17}$$

for $x \in \mathbf{R}$. Then, its α-cuts are

$$\widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X})_\alpha = [\mathrm{AVaR}_p^\nu(\tilde{X}_\alpha^-), \mathrm{AVaR}_p^\nu(\tilde{X}_\alpha^+)]. \tag{18}$$

for $\alpha \in [0,1]$. Hence, we obtain the following lemma from Lemma 2 and (18) in the same way as [34].

Lemma 4 *Let $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{X}}$. Then, the following (i)–(iv) hold:*

(i) *If $\tilde{X} \preceq \tilde{Y}$, then $\widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X}) \preceq \widetilde{\mathrm{AVaR}}_p^\nu(\tilde{Y})$.*

(ii) $\widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X} + \tilde{c}) = \widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X}) + \tilde{c}$ *for $\tilde{c} \in \mathcal{N}$.*

(iii) $\widetilde{\mathrm{AVaR}}_p^\nu(c\tilde{X}) = c\,\widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X})$ *for $c \in \mathbf{R}_+$.*

(iv) $\widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X} + \tilde{Y}) \succeq \widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X}) + \widetilde{\mathrm{AVaR}}_p^\nu(\tilde{Y})$.

We can also extend value-at-risk VaR_p and average value-at-risk AVaR_p in the same way [34]. For a coherent risk measure ρ, its extended measure for a fuzzy random variable $\tilde{X} \in \tilde{\mathcal{X}}$ is a fuzzy number

$$\tilde{\rho}(\tilde{X})(x) = \sup_{X \in \mathcal{X} : \rho(X) = x} \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)) \tag{19}$$

for $x \in \mathbf{R}$. Its α-cut is given by $\tilde{\rho}(\tilde{X})_\alpha = [\rho(\tilde{X}_\alpha^+), \rho(\tilde{X}_\alpha^-)]$, and this extended measure $\tilde{\rho}(\cdot)$ has the following properties from Definition 2 and [34].

Lemma 5 *Let $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{X}}$. Then, the following (i)–(iv) hold:*

(i) *If $\tilde{X} \preceq \tilde{Y}$, then $\tilde{\rho}(\tilde{X}) \succeq \tilde{\rho}(\tilde{Y})$.*

(ii) $\tilde{\rho}(\tilde{X} + \tilde{c}) = \tilde{\rho}(\tilde{X}) - \tilde{c}$ *for $\tilde{c} \in \mathcal{N}$.*

(iii) $\tilde{\rho}(c\tilde{X}) = c\,\tilde{\rho}(\tilde{X})$ *for $c \in \mathbf{R}_+$.*

(iv) $\tilde{\rho}(\tilde{X} + \tilde{Y}) \preceq \tilde{\rho}(\tilde{X}) + \tilde{\rho}(\tilde{Y})$.

5 Estimation of fuzziness with λ-mean functions and evaluation weights

Defuzzification has been studied by many researchers. In this paper, the fuzziness of fuzzy numbers and fuzzy random variables is estimated by *λ-mean functions*

$$[x, y] \mapsto \lambda \cdot x + (1 - \lambda) \cdot y \tag{20}$$

with $\lambda \in [0,1]$ and *evaluation weights* $w(\alpha)$ for closed intervals $[x, y]$ [32, 33]. Hence, λ is called the *pessimistic index* if $\lambda = 1$, and it is called the *optimistic index* if $\lambda = 0$ [9]. A defuzzification of a fuzzy number $\tilde{n} \in \mathcal{N}$ with λ-mean and weight $w(\alpha)$ is given by

$$E^\lambda(\tilde{n}) = \frac{\int_0^1 (\lambda \cdot \tilde{n}_\alpha^- + (1-\lambda) \cdot \tilde{n}_\alpha^+)\, w(\alpha)\, \mathrm{d}\alpha}{\int_0^1 w(\alpha)\, \mathrm{d}\alpha}, \tag{21}$$

where α-cuts of $\tilde{n}$ are closed intervals $\tilde{n}_\alpha = [\tilde{n}_\alpha^-, \tilde{n}_\alpha^+]$. Here, $w(\alpha)$ is called the *possibility evaluation weight* if $w(\alpha) = 1$ for $\alpha \in [0,1]$, and it is also called the *necessity evaluation weight* if $w(\alpha) = 1 - \alpha$ for $\alpha \in [0,1]$. This approach is corresponding to the idea in Drago and Ridella [6]. Then, we have the following lemma from [32, Theorem 1].

Lemma 6 *Let $\lambda \in [0,1]$. $E^\lambda(\cdot)$ be monotonically increasing, positively homogeneous and additive on $\mathcal{N}$.*

For a fuzzy random variable $\tilde{X} \in \tilde{\mathcal{X}}$, *the expectation of the mean* $E(E^\lambda(\tilde{X}))$ is a real number

$$E(E^\lambda(\tilde{X})) = E\left(\frac{\int_0^1 (\lambda \cdot \tilde{X}_\alpha^- + (1-\lambda) \cdot \tilde{X}_\alpha^+)\, w(\alpha)\, d\alpha}{\int_0^1 w(\alpha)\, \mathrm{d}\alpha}\right). \tag{22}$$

Hence, the following lemma holds from [32, Corollary 1].

Lemma 7 [32, 33] *Let $\lambda \in [0,1]$. $E(E^\lambda(\cdot))$ has the following properties (i)–(iii):*

(i) $E(E^\lambda(\tilde{X})) = E^\lambda(\tilde{E}(\tilde{X}))$ *for $\tilde{X} \in \tilde{\mathcal{X}}$.*

(ii) $E(E^\lambda(\tilde{n})) = E^\lambda(\tilde{n})$ *and* $E(E^\lambda(X)) = E(X)$ for $\tilde{n} \in \mathcal{N}$ *and $X \in \mathcal{X}$.*

(iii) *$E(E^\lambda(\cdot))$ is monotonically increasing, positively homogeneous and additive on $\tilde{\mathcal{X}}$.*

Let $\tilde{\mathcal{X}}_a$ be a family of fuzzy random variables $\tilde{X} \in \tilde{\mathcal{X}}$ for which there exist a random variable $X \in \mathcal{X}$ and a fuzzy number $\tilde{n} \in \mathcal{N}$ such that

$$\tilde{X}(\omega)(x) = 1_{\{X(\omega)\}}(x) + \tilde{n}(x) \tag{23}$$

for $\omega \in \Omega$ and $x \in \mathbf{R}$, where $1_{\{\cdot\}}$ denotes the characteristic function of a singleton. Lemma 7(i) implies that the expectation $E(\cdot)$ and the mean $E^\lambda(\cdot)$ are exchangeable. On the other hand, we obtain the following exchangeabilities on $\tilde{\mathcal{X}}_a$.

Proposition 1 *Let $\lambda \in [0,1]$ and $p \in (0,1]$. For weighted average value-at-risks and coherent risk measures, the following equations hold:*

$$E^\lambda(\widetilde{\mathrm{AVaR}}_p^\nu(\tilde{X})) = \mathrm{AVaR}_p^\nu(E^\lambda(\tilde{X})), \tag{24}$$

$$E^{1-\lambda}(\tilde{\rho}(\tilde{X})) = \rho(E^\lambda(\tilde{X})) \tag{25}$$

for fuzzy random variables $\tilde{X} \in \tilde{\mathcal{X}}_a$.

Proof Let $\lambda \in [0,1]$ and $p \in (0,1]$. From (23), for $\tilde{X} \in \tilde{\mathcal{X}}_a$ there exist a random variable $X \in \mathcal{X}$ and a fuzzy number $\tilde{n} \in \mathcal{N}$ such that $\tilde{X}(\omega)(x) = 1_{\{X(\omega)\}}(x) + \tilde{n}(x)$ for $\omega \in \Omega$ and $x \in \mathbf{R}$. Then, its α-cuts are

$$\tilde{X}_\alpha = [\tilde{X}_\alpha^-, \tilde{X}_\alpha^+] = [X + \tilde{n}_\alpha^-, X + \tilde{n}_\alpha^+] \tag{26}$$

for $\alpha \in [0,1]$. Together with from (18), (21) and Lemma 2(iii), we get

$$\begin{aligned}
&E^\lambda(\widetilde{\mathrm{AVaR}}_p^v(\tilde{X}))) \\
&= \frac{\int_0^1 (\lambda \cdot \mathrm{AVaR}_p^v(\tilde{X}_\alpha^-) + (1-\lambda)\cdot \mathrm{AVaR}_p^v(\tilde{X}_\alpha^+))\, w(\alpha)\, \mathrm{d}\alpha}{\int_0^1 w(\alpha)\,\mathrm{d}\alpha} \\
&= \frac{\int_0^1 (\lambda \cdot \mathrm{AVaR}_p^v(X+\tilde{n}_\alpha^-) + (1-\lambda)\cdot \mathrm{AVaR}_p^v(X+\tilde{n}_\alpha^+))\, w(\alpha)\, \mathrm{d}\alpha}{\int_0^1 w(\alpha)\,\mathrm{d}\alpha} \\
&= \frac{\int_0^1 (\lambda \cdot (\mathrm{AVaR}_p^v(X)+\tilde{n}_\alpha^-) + (1-\lambda)\cdot (\mathrm{AVaR}_p^v(X)+\tilde{n}_\alpha^+))\, w(\alpha)\, \mathrm{d}\alpha}{\int_0^1 w(\alpha)\,\mathrm{d}\alpha} \\
&= \mathrm{AVaR}_p^v(X) + \frac{\int_0^1 (\lambda\cdot\tilde{n}_\alpha^- + (1-\lambda)\cdot\tilde{n}_\alpha^+)\, w(\alpha)\, d\alpha}{\int_0^1 w(\alpha)\,\mathrm{d}\alpha} \\
&= \mathrm{AVaR}_p^v(X) + \tilde{E}^\lambda(\tilde{n}) \\
&= \mathrm{AVaR}_p^v(X + E^\lambda(\tilde{n})) \\
&= \mathrm{AVaR}_p^v(E^\lambda(\tilde{X})).
\end{aligned}$$

Thus, (24) holds. We can check (25) similarly. Therefore, this lemma holds. □

In Proposition 1, we note $\mathrm{AVaR}_p^v(E^\lambda(\cdot))$ is non-decreasing and $\rho(E^\lambda(\cdot))$ is non-increasing. Let $\lambda \in [0,1]$. Finally, from [32], we define a covariance by

$$\begin{aligned}&Cov(E^\lambda(\tilde{X}), E^\lambda(\tilde{Y})) = E\big((E^\lambda(\tilde{X}) - E(E^\lambda(\tilde{X})))\\ &(E^\lambda(\tilde{Y}) - E(E^\lambda(\tilde{Y})))\big)\end{aligned} \tag{27}$$

for fuzzy random variables $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{X}}$. We have the following lemma from [32, Theorems 4 and 5] .

Lemma 8 *Let $\lambda \in [0,1]$. For fuzzy numbers $\tilde{n}, \tilde{m} \in \mathcal{N}$, fuzzy random variables $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{X}}$ and $c \in \mathbf{R}_+$, the following (i)–(iii) hold:*

(i) $Cov(E^\lambda(c\tilde{X}), E^\lambda(c\tilde{Y})) = c^2 Cov(E^\lambda(\tilde{X}), E^\lambda(\tilde{Y}))$.
(ii) $Cov(E^\lambda(\tilde{X}), E^\lambda(\tilde{n})) = Cov(E^\lambda(\tilde{n}), E^\lambda(\tilde{X})) = 0$.
(iii) $Cov(E^\lambda(\tilde{X}+\tilde{n}), E^\lambda(\tilde{Y}+\tilde{m})) = Cov(E^\lambda(\tilde{X}), E^\lambda(\tilde{Y}))$.

6 Portfolio optimization with fuzzy random variables

Let a probability $p \in (0,1]$ and $\lambda \in [0,1]$. Let v be the risk spectrum in Lemma 3. Let ρ be the coherent risk measure given by $\rho = -\mathrm{AVaR}_p^v$, where AVaR_p^v is the weighted average value-at-risk (3). Let an expiration date T be a positive integer. In this paper, we deal with a portfolio model regarding n stocks as risky assets, where n is a positive integer. For $i = 1,2,\ldots,n$, $\{S_t^i\}_{t=0}^T$ denotes an *stock price process* where the initial stock price S_0^i is a positive number. Let R_t^i be the *rate of return* at time t, which satisfies $1 + R_t^i \geq 0$ and

$$S_t^i = S_{t-1}^i(1 + R_t^i) \tag{28}$$

for $t = 1,2,\ldots,T$. In this paper, we discuss a case where the rates of return have fuzziness, i.e., they are given by a sequence of fuzzy random variables $\{\tilde{R}_t^i\}_{t=1}^T \subset \tilde{\mathcal{X}}_a$. Hence, we assume

$$1 + \tilde{R}_t^i \succeq 0 \tag{29}$$

for all $t = 1,2,\ldots,T$ and $i = 1,2,\ldots,n$. Define a set of vectors $\mathcal{W}_t = \{(w_t^1, w_t^2, \ldots, w_t^n) \in \mathbf{R}^n \mid \sum_{i=1}^n w_t^i = 1$ and $w_t^i \geq 0$ for $i = 1,2,\ldots,n\}$. As trading strategies, we use *portfolio weight vectors* $(w_t^1, w_t^2, \ldots, w_t^n) \in \mathcal{W}_t$. The rate of return with a portfolio weight vector $(w_t^1, w_t^2, \ldots, w_t^n) \in \mathcal{W}_t$ is given by

$$\tilde{R}_t = \sum_{i=1}^n w_t^i \tilde{R}_t^i. \tag{30}$$

Let $t = 1,2,\ldots,T$ and $\lambda \in [0,1]$. Let the mean and the covariance of the rates of return $\tilde{R}_t^i$, respectively, be

$$\mu_t^i = E(E^\lambda(\tilde{R}_t^i)), \tag{31}$$

$$\sigma_t^{ij} = Cov(E^\lambda(\tilde{R}_t^i), E^\lambda(\tilde{R}_t^j)) = E((E^\lambda(\tilde{R}_t^i) - \mu_t^i)(E^\lambda(\tilde{R}_t^j) - \mu_t^j)) \tag{32}$$

for $i,j = 1,2,\ldots,n$. Let a vector of rates of return, a variance–covariance matrix and real numbers as follows:

$$\mu_t = \begin{bmatrix} \mu_t^1 \\ \mu_t^2 \\ \vdots \\ \mu_t^n \end{bmatrix}, \Sigma_t = \begin{bmatrix} \sigma_t^{11} & \sigma_t^{12} & \cdots & \sigma_t^{1n} \\ \sigma_t^{21} & \sigma_t^{22} & \cdots & \sigma_t^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_t^{n1} & \sigma_t^{n2} & \cdots & \sigma_t^{nn} \end{bmatrix}, \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{33}$$

$A_t = \mathbf{1}^\mathrm{T}\Sigma_t^{-1}\mathbf{1}, B_t = \mathbf{1}^\mathrm{T}\Sigma_t^{-1}\mu_t, C_t = \mu_t^\mathrm{T}\Sigma_t^{-1}\mu_t$ and $\Delta_t = A_tC_t - B_t^2$, where T denotes the transpose of a vector. We assume the determinant of the variance–covariance matrix Σ_t is not zero, then there exists its inverse positive definite matrix Σ_t^{-1}, and we have $A_t > 0$ and $\Delta_t > 0$. These assumptions are natural, and they can be realized easily by taking care of the combinations of stocks. From Lemmas 6, 7 and 8, for a portfolio $w_t = (w_t^1, w_t^2, \ldots, w_t^n) \in \mathcal{W}_t$, the expectation and the variance regarding $\tilde{R}_t = \sum_{i=1}^n w_t^i \tilde{R}_t^i$ are calculated as follows:

$$E(E^\lambda(\tilde{R}_t)) = \sum_{i=1}^n w_t^i E(E^\lambda(\tilde{R}_t^i)) = \sum_{i=1}^n w_t^i \mu_t^i, \tag{34}$$

$$V(E^{\lambda}(\tilde{R}_t)) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_t^i w_t^j Cov(E^{\lambda}(\tilde{R}_t^i), E^{\lambda}(\tilde{R}_t^j)) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_t^i w_t^j \sigma_t^{ij}. \tag{35}$$

From (8), (34) and (35), we have value-at-risk

$$\mathrm{VaR}_p(E^{\lambda}(\tilde{R}_t)) = \sum_{i=1}^{n} w_t^i \mu_t^i + \kappa(p)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_t^i w_t^j \sigma_t^{ij}}. \tag{36}$$

From (3), (36) and Proposition 1, the value-at-risk $E^{\lambda}(\widetilde{\mathrm{AVaR}}_p^{\nu}(\tilde{R}_t))$ has the following representation:

$$E^{\lambda}(\widetilde{\mathrm{AVaR}}_p^{\nu}(\tilde{R}_t)) = \sum_{i=1}^{n} w_t^i \mu_t^i + \kappa^{\nu}(p)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_t^i w_t^j \sigma_t^{ij}} \tag{37}$$

with constant

$$\kappa^{\nu}(p) = \frac{\int_0^p \kappa(q)\nu(q)\,\mathrm{d}q}{\int_0^p \nu(q)\,\mathrm{d}q}, \tag{38}$$

where $\kappa(\cdot)$ is given in (8). One of the sufficient conditions for (8) and (36) is what the rates of return R_t^i $(i = 1, 2, \ldots, n)$ have normal distributions. From Lemma 1 and Proposition 1, the mean of a risk measure $\tilde{\rho}(\tilde{R}_t) = -\widetilde{\mathrm{AVaR}}_p^{\nu}(\tilde{R}_t)$ can be evaluated as

$$E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t)) = -E^{\lambda}(\widetilde{\mathrm{AVaR}}_p^{\nu}(\tilde{R}_t)) = -\sum_{i=1}^{n} w_t^i \mu_t^i - \kappa^{\nu}(p)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_t^i w_t^j \sigma_t^{ij}}. \tag{39}$$

By mathematical programming, this paper discusses a portfolio problem to minimize the risk values (39) in three steps. Let a constant $\gamma \in \mathbf{R}$. First, we deal with the following classical problem.

Problem 1 Minimize the variance

$$V(E^{\lambda}(\tilde{R}_t)) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_t^i w_t^j \sigma_t^{ij} \tag{40}$$

with respect to portfolios $w_t = (w_t^1, w_t^2, \ldots, w_t^n) \in \mathcal{W}_t(\gamma)$, where $\mathcal{W}_t(\gamma) = \{(w_t^1, w_t^2, \ldots, w_t^n) \in \mathcal{W}_t \mid \sum_{i=1}^{n} w_t^i \mu_t^i = \gamma\}$.

From the classical results in quadratic programming, we obtain the following lemma [35, 36].

Lemma 9 *The optimal portfolio in Problem 1 is given by*

$$w_t^{\circ} = \xi^{\circ}\Sigma_t^{-1}\mathbf{1} + \eta^{\circ}\Sigma_t^{-1}\mu_t \tag{41}$$

and then the corresponding variance is

$$\min_{w_t \in \mathcal{W}_t(\gamma)} V(E^{\lambda}(\tilde{R}_t)) = \frac{A_t\gamma^2 - 2B_t\gamma + C_t}{\Delta_t}, \tag{42}$$

where

$$\xi^{\circ} = \frac{C_t - B_t\gamma}{\Delta_t} \quad \text{and} \quad \eta^{\circ} = \frac{A_t\gamma - B_t}{\Delta_t}. \tag{43}$$

The solution w in Lemma 9 is called a *minimal risk portfolio* [21, 24]. Next for a constant γ we discuss the following risk minimization portfolio problem.

Problem 2 Minimize the risk values of the rate of return

$$E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t)) = -\sum_{i=1}^{n} w_t^i \mu_t^i - \kappa^{\nu}(p)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_t^i w_t^j \sigma_t^{ij}} \tag{44}$$

with respect to portfolios $w_t = (w_t^1, w_t^2, \ldots, w_t^n) \in \mathcal{W}_t(\gamma)$.

Then, the following result is trivial from Lemma 9.

Lemma 10 *Let $\kappa^{\nu}(p)$ satisfy $\kappa^{\nu}(p) < -\sqrt{\Delta_t/A_t}$. Then, the optimal risk value in Problem 2 is*

$$\inf_{w_t \in \mathcal{W}_t(\gamma)} E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t)) = -\gamma - \kappa^{\nu}(p)\sqrt{\frac{A_t\gamma^2 - 2B_t\gamma + C_t}{\Delta_t}}. \tag{45}$$

Hence, the function (45) has the following properties from [36, Theorem 4.1]

Lemma 11 *Let $\kappa^{\nu}(p)$ satisfy $\kappa^{\nu}(p) < -\sqrt{\Delta_t/A_t}$. Then, a real-valued function*

$$\gamma(\in \mathbf{R}) \mapsto -\gamma - \kappa^{\nu}(p)\sqrt{\frac{A_t\gamma^2 - 2B_t\gamma + C_t}{\Delta_t}} \tag{46}$$

is convex and it has the minimum

$$-\frac{B_t}{A_t} + \frac{\sqrt{A_t\kappa^{\nu}(p)^2 - \Delta_t}}{A_t} \tag{47}$$

at

$$\gamma_t^* = \frac{B_t}{A_t} + \frac{\Delta}{A_t\sqrt{A_t\kappa^{\nu}(p)^2 - \Delta_t}}. \tag{48}$$

Finally, we discuss the following minimization problem of $E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t))$.

Problem 3 Minimize the risk values

$$E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t)) = -\sum_{i=1}^{n} w_t^i \mu_t^i - \kappa^v(p)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_t^i w_t^j \sigma_t^{ij}} \tag{49}$$

with respect to portfolios $w_t = (w_t^1, w_t^2, \ldots, w_t^n) \in \mathcal{W}_t$.

Because

$$\inf_{w_t \in \mathcal{W}_t} E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t)) = \inf_{\gamma}\left(\inf_{w_t \in \mathcal{W}_t(\gamma)} E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t))\right), \tag{50}$$

from Lemmas 9 and 11 we arrive at the following analytical solutions for Problem 3 in a similar way to [35].

Theorem 1 *Let $\kappa^v(p)$ satisfy $\kappa^v(p) < -\sqrt{\Delta_t/A_t}$.*

(i) *The minimum risk value in Problem 3 is*

$$\rho_t^* = \inf_{w_t \in \mathcal{W}_t} E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t)) = -\frac{B_t}{A_t} + \frac{\sqrt{A_t \kappa^v(p)^2 - \Delta_t}}{A_t}, \tag{51}$$

and then the corresponding expected rate of return is

$$\gamma_t^* = \frac{B_t}{A_t} + \frac{\Delta_t}{A_t\sqrt{A_t \kappa^v(p)^2 - \Delta_t}}. \tag{52}$$

(ii) *The optimal portfolio of Problem 3 is given by*

$$w_t^* = \xi_t^* \Sigma_t^{-1}\mathbf{1} + \eta_t^* \Sigma_t^{-1}\mu_t, \tag{53}$$

where

$$\xi_t^* = \frac{C_t - B_t\gamma_t^*}{\Delta_t} \quad \text{and} \quad \eta_t^* = \frac{A_t\gamma_t^* - B_t}{\Delta_t}. \tag{54}$$

(iii) *The portfolio (53) satisfies $w_t^* \geq \mathbf{0}$ if $\Sigma_t^{-1}\mathbf{1} \geq \mathbf{0}$ and $\Sigma_t^{-1}\mu_t \geq \mathbf{0}$ for $t = 1, 2, \ldots, T$, where $\mathbf{0}$ denotes the zero vector.*

7 Numerical examples

In this section, we give a few examples to understand the results in the previous sections. Yoshida [39] has studied the relations between various utility functions and their risk premia. In Examples 1 and 2, we discuss risk neutral utility functions and the risk averse utility function in Sect. 3, and we compare the results.

Example 1 Let a domain $I = \mathbf{R}$ and let f be a *risk neutral utility function*

$$f(x) = ax + b \tag{55}$$

for $x \in \mathbf{R}$ with constants $a(>0)$ and $b(\in \mathbf{R})$. From [39, Example 1], its risk spectrum in Lemma 3 is given by $v(p) = 1$, and then the corresponding coherent risk measure is the *average value-at-risk* (2). Therefore, we have

$$E^{\lambda}(\widetilde{\mathrm{AVaR}}_p^v(\tilde{X})) = \mathrm{AVaR}_p(E^{\lambda}(\tilde{X})) = \frac{1}{p}\int_0^p \mathrm{VaR}_q(E^{\lambda}(\tilde{X}))\,\mathrm{d}q \tag{56}$$

for $\tilde{X} \in \tilde{\mathcal{X}}$ and $p \in (0,1]$. We can find this portfolio optimization in Yoshida [38, Sect. 6(i)].

Example 2 Let a domain $I = \mathbf{R}$ and let a *risk averse exponential utility function*

$$f(x) = \frac{1 - e^{-\tau x}}{\tau} \tag{57}$$

for $x \in \mathbf{R}$ with a constant $\tau(>0)$. Let $\mathcal{X}$ be a family of random variables X which have normal distribution functions. Define the cumulative distribution function $\Phi : \mathbf{R} \to (0,1)$ of the standard normal distribution by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\,\mathrm{d}z \tag{58}$$

for $x \in \mathbf{R}$, and define an increasing function $\kappa : (0,1) \mapsto \mathbf{R}$ by its inverse function

$$\kappa(p) = \Phi^{-1}(p) \tag{59}$$

for probabilities $p \in (0,1)$. Then, the value-at-risk satisfies (8) with a mean $\mu = E(X)$ and a standard deviation $\sigma = \sigma(X)$. Suppose there exists a distribution function $\psi : \mathbf{R} \times (0,\infty) \mapsto [0,\infty)$ such that

$$\psi(\mu,\sigma) = \phi(\mu)\cdot\frac{2^{1-n/2}}{\Gamma(n/2)}\,\sigma^{n-1}e^{-\frac{\sigma^2}{2}} \tag{60}$$

for $(\mu,\sigma) \in \mathbf{R} \times [0,\infty)$, where $\phi(\mu)$ is some probability distribution, $\Gamma(\cdot)$ is a gamma function and $\frac{2^{1-n/2}}{\Gamma(n/2)}\,\sigma^{n-1}e^{-\frac{\sigma^2}{2}}$ is a chi distribution with degree of freedom n. From [39, Example 2], the component function (10) for the risk spectrum v in Lemma 3 is reduced to

$$C(p) = \frac{1}{p}\cdot\frac{\int_0^{\infty}\left(1 - \frac{1}{\frac{1}{p}\int_0^p e^{\tau\sigma(\kappa(p)-\kappa(q))}\,\mathrm{d}q}\right)\sigma^n e^{-\frac{\sigma^2}{2}}\,\mathrm{d}\sigma}{\int_0^{\infty}\log\left(\frac{1}{p}\int_0^p e^{\tau\sigma(\kappa(p)-\kappa(q))}\,\mathrm{d}q\right)\sigma^n e^{-\frac{\sigma^2}{2}}\,\mathrm{d}\sigma}. \tag{61}$$

Figures 1 and 2 illustrate utility functions $f(x)$ and the corresponding risk spectra $v(p)$. We give the rates of return $\tilde{R}_t^i \in \tilde{\mathcal{X}}_a$ by the following fuzzy random variables

$$\tilde{R}_t^i(\omega)(\cdot) = 1_{\{R_t^i(\omega)\}}(\cdot) + \tilde{a}_t^i(\cdot) \tag{62}$$

for $\omega \in \Omega$, where R_t^i has a normal distribution with the mean value $E(R_t^i)$ and $\tilde{a}_t^i$ is a triangle-type fuzzy number

$$\tilde{a}_t^i(x) = \max\{1 - |x|/c_t^i, 0\} \tag{63}$$

for $x \in \mathbf{R}$ with a positive number c_t^i. Here, we give a simple example to illustrate our idea. Let $n = 4$ be the number of assets. Take the expected rate of return and a variance–covariance matrix as Table 1. We deal with a case of the *pessimistic index* ($\lambda = 1$) and the *necessity evaluation weight* ($w(\alpha) = 1 - \alpha$). For example, in a case of risk probability 5%, i.e., $p = 0.05$, in the normal distribution and utility function $f(x) = 1 - e^{-x}$ with $\tau = 1$ in (57), we can easily calculate $A_t = 13.5861 > 0$, $\Delta_t = 0.0112653 > 0$ and $\kappa^v(p) < -\sqrt{\Delta_t/A_t} = -0.0287955$ for all $p \in (0, 1]$. From Theorem 1, we easily obtain the optimal portfolio $w_t^* = (w_t^1, w_t^2, w_t^3, w_t^4) = (0.247093, 0.281828, 0.304902, 0.166177)$ for Problem 3, and then the expected rate of return is $\gamma_t^* = 0.0713242$ and the minimum risk value is $\rho_t^* = \inf_{w_t \in \mathcal{W}_t} E^{1-\lambda}(\tilde{\rho}(\tilde{R}_t)) = -\sup_{w_t \in \mathcal{W}_t} E^{\lambda}(\widetilde{\mathrm{AVaR}}_p^v(\tilde{R}_t)) = 0.551907$.

For $p = 0.01$, Table 2 shows the expected rates of return γ_t^* and the minimum risk values ρ_t^* in case of pessimistic index $\lambda = 1$ and necessity evaluation $w(\alpha) = 1 - \alpha$ and in case of optimistic index $\lambda = 0$ and possibility evaluation $w(\alpha) = 1$. Then, we can observe $0.071308 \leq \gamma_t^* \leq 0.082974$ and $0.678478 \leq \rho_t^* \leq 0.666811$ in Example 2 ($\tau = 1$), and this range is depending on decision maker's selection of pessimistic–optimistic index λ and possibility–necessity weight $w(\alpha)$, which are decided by his certainty about information in the stock market.

It is well known that the degree of decision maker's risk averse attitude is represented by Arrow's *absolute risk averse indexes* $-f''/f'$, which is calculated as $-f''/f' = 0$ in Example 1 with risk neutral utilities and which follows $-f''/f' = \tau(>0)$ in Example 2 with risk averse utilities [3]. Table 3 implies the comparison of the expected rates of return γ_t^* and the minimum risk values ρ_t^* for utility functions $f(x)$ and their risk averse indexes $-f''/f'$. In Table 3, the minimum risk value ρ_t^* becomes larger and the expected rate of return γ_t^* is increasing a little when the risk averse index is larger. These data reflect the risk aversity of the utility functions because weighted average value-at-risks AVaR_p^v with risk spectrum v are taking over the decision maker's risk averse behavior. Figures 3 and 4 illustrate the risk values ρ_t^* and the expected rates of return γ_t^* for Examples 1 and 2 ($\tau = 1, 2$). In Fig. 3, we can observe the expected rate of return γ_t^* of Example 1 increases rapidly to infinity when p approaches to 1; however, γ_t^* of Example 2 ($\tau = 1, 2$) remain stable. The reason comes from that the minimum risk value of Example 1 gets close and crosses the line $\rho_t^* = 0$, which implies the no risk line (Fig. 4). These drastic changes of graphs γ_t^* and ρ_t^* in Example 1 look abnormal. Thus, coherent risk measure given by average value-at-risk $\rho = -\mathrm{AVaR}_p$, which corresponds to risk neutral utility function in Example 1, gives us reasonable results only if probability p is small. On the other hand, the coherent risk measures $\rho = -\mathrm{AVaR}_p^v$ derived from risk averse utility functions in Example 2 bring us stable and reasonable results for any positive probability p in portfolio optimization.

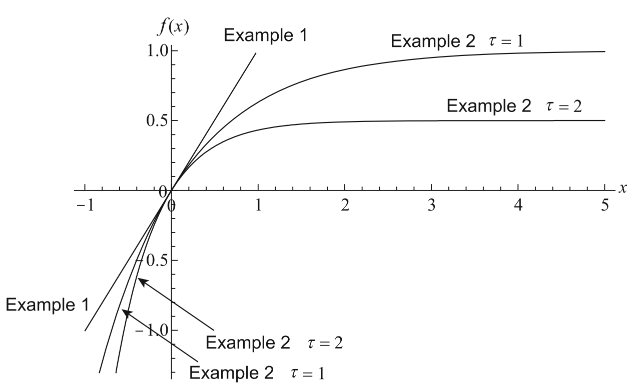

Fig. 1 Utility functions $f(x)$

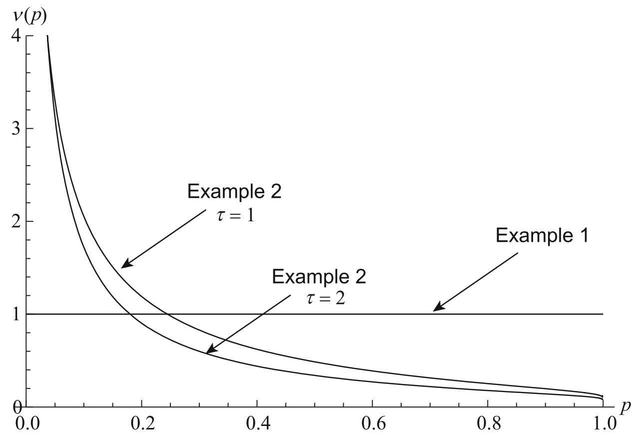

Fig. 2 Risk spectra $v(p)$

Table 1 Rate of return μ_t with fuzzy factor and variance–covariance matrix Σ_t

μ_t^i	$E(R_t^i)$	c_t^i		
$i = 1$	0.09	0.010		
$i = 2$	0.07	0.009		
$i = 3$	0.08	0.008		
$i = 4$	0.07	0.007		
σ_t^{ij}	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	0.38	− 0.06	− 0.05	0.08
$i = 2$	− 0.06	0.34	− 0.06	0.06
$i = 3$	− 0.05	− 0.06	0.36	− 0.04
$i = 4$	0.08	0.06	− 0.04	0.29

Table 2 The expected rates of return γ_t^* and the minimum risk values ρ_t^* for pessimistic–optimistic indexes λ and possibility–necessity weights $w(\alpha)$ $(p = 0.01)$

Risk measure $\tilde{\rho}$	$-\widetilde{\mathrm{AVaR}}_p^v$ $(\tau = 1)$		$-\widetilde{\mathrm{AVaR}}_p^v$ $(\tau = 2)$	
Pess./Opti. & Nec./Poss.	Pess. & Nec.	Opti. & Poss.	Pess. & Nec.	Opti. & Poss.
Expected rate of return γ_t^*	0.071308	0.082974	0.071305	0.082971
Minimum risk value ρ_t^*	0.678478	0.666811	0.707726	0.696059

Table 3 The expected rates of return γ_t^* and the minimum risk values ρ_t^* for utility functions $f(x)$ and their risk averse indexes $-f''/f'$ $(p = 0.05)$

Risk measure $\tilde{\rho}$	$-\widetilde{\mathrm{VaR}}_p$	$-\widetilde{\mathrm{AVaR}}_p$	$-\widetilde{\mathrm{AVaR}}_p^v$ $(\tau = 1)$	$-\widetilde{\mathrm{AVaR}}_p^v$ $(\tau = 2)$
Utility function $f(x)$	–	x	$1 - e^{-x}$	$(1 - e^{-2x})/2$
Risk averse index $-f''/f'$	–	0	1	2
Expected rate of return γ_t^*	0.071363	0.071335	0.071324	0.071314
Minimum risk value ρ_t^*	0.374956	0.502218	0.551907	0.624703

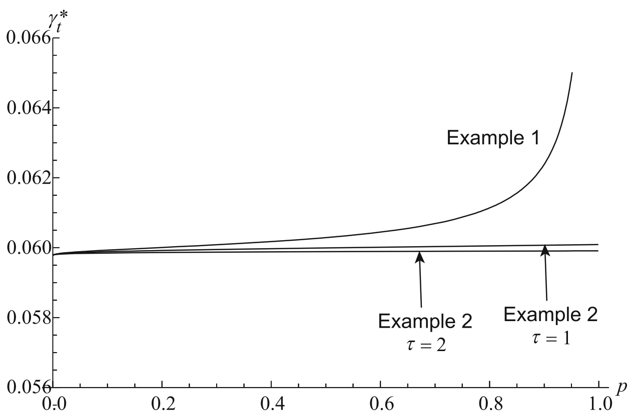

Fig. 3 The expected rates of return γ_t^*

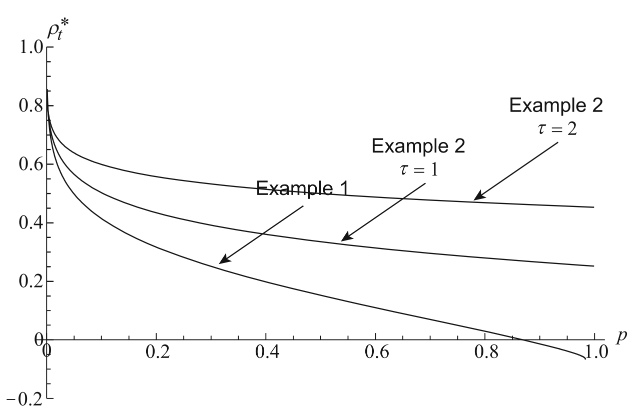

Fig. 4 The minimum risk values ρ_t^*

8 Concluding remarks

In Sect. 7, we have estimated fuzzy random variables not only by pessimistic index $\lambda = 1$ and necessity evaluation $w(\alpha) = 1 - \alpha$ but also by optimistic index $\lambda = 0$ and possibility evaluation $w(\alpha) = 1$ (Table 2). The parameters should be chosen based on decision maker's philosophy in investigation and his observation of the stock market.

Decision maker's utility function (57) is characterized by the parameter τ, which coincides its risk averse index $-f''/f'$, i.e., the degree of his risk averse attitude (Table 3). The parameter τ should be revised by repetition of trial and error as an important factor representing his decision making attitude. In such a way, the decision maker can use a risk criterion based on his utility f quickly and he can make asset management stable.

Using the risk spectrum v in Lemma 3, we can incorporate the decision maker's risk averse attitude f into coherent risk measures. As we have seen in Figs. 3 and 4, the coherent risk measures with the risk spectrum v bring us reasonable estimation in portfolio optimization not only for small probabilities p but also large probabilities p. This approach will be applicable to subjective risk measurement for both investment and speculation in finance and management. In the next topic, we will need to investigate dynamic portfolio optimization models using the coherent risk measures with the risk spectrum v.

Acknowledgements This research is supported from JSPS KAKENHI Grant Number JP 16K05282.

References

1. Acerbi C (2002) Spectral measures of risk: a coherent representation of subjective risk aversion. J Bank Finance 26:1505–1518
2. Adam A, Houkari H, Laurent JP (2008) Spectral risk measures and portfolio selection. J Bank Finance 32:1870–1882
3. Arrow KL (1971) Essays in the theory of risk-bearing. Markham, Chicago
4. Artzner P, Delbaen F, Eber JM, Heath D (1999) Coherent measures of risk. Math Finance 9:203–228
5. Bellman RE, Zadeh LA (1970) Decision-making in a fuzzy environment. Manage Sci Ser B 17:141–164
6. Drago GP, Ridella S (1999) Possibility and necessity pattern classification using an interval arithmetic perceptron. Neural Comput Appl 8:40–52

7. Emmer S, Kratz M, Tasche D (2015) What is the best risk measure in practice? A comparison of standard measures. J Risk 18:31–60
8. Fang Y, Lai KK, Wang S (2008) Fuzzy portfolio optimization. Lecture Notes in Economics and Mathematical Systems, vol 609. Springer, Heidelberg
9. Fortemps P, Roubens M (1996) Ranking and defuzzification methods based on area compensation. Fuzzy Sets Syst 82:319–330
10. Guo H, Pedrycz W, Liu X (2018) Fuzzy time series forecasting based on axiomatic fuzzy set theory. Neural Comput Appl. https://doi.org/10.1007/s00521-017-3325-9
11. Hasuike T, Katagiri H, Ishii H (2009) Portfolio selection problems with random fuzzy variable returns. Fuzzy Sets Syst 160:2579–2596
12. Jorion P (2006) Value at risk: the new benchmark for managing financial risk. McGraw-Hill, New York
13. Katagiri H, Sakawa M, Kato K, Nishizaki I (2008) Interactive multiobjective fuzzy random linear programming: maximization of possibility and probability. Eur J Oper Res 188:530–539
14. Kodogiannis V, Lolis A (2002) Forecasting financial time series using neural network and fuzzy system-based techniques. Neural Comput Appl 11:90–102
15. Kruse R, Meyer KD (1987) Statistics with vague data. Reidel Publishing Company, Dortrecht
16. Kusuoka S (2001) On law-invariant coherent risk measures. Adv Math Econ 3:83–95
17. Kwakernaak H (1978) Fuzzy random variables: I. Definitions and theorem. Inf Sci 15:1–29
18. Li J, Xu J (2013) Multi-objective portfolio selection model with fuzzy random returns and a compromise approach-based genetic algorithm. Inf Sci 220:507–521
19. Markowitz H (1990) Mean–variance analysis in portfolio choice and capital markets. Blackwell, Oxford
20. Moussa AM, Kamdem JS, Terraza M (2014) Fuzzy value-at-risk and expected shortfall for portfolios with heavy-tailed returns. Econ Model 39:247–256
21. Pliska SR (1997) Introduction to mathematical finance: discrete-time models. Blackwell, New York
22. Puri ML, Ralescu DA (1986) Fuzzy random variables. J Math Anal Appl 114:409–422
23. Rockafellar RT, Uryasev S (2000) Optimization of conditional value-at-risk. J Risk 2:21–41
24. Ross SM (1999) An introduction to mathematical finance. Cambridge University Press, Cambridge
25. Sadati MEH, Doniavi A (2014) Optimization of fuzzy random portfolio selection by implementation of harmony search algorithm. Int J Eng Trends Technol 8:60–64
26. Sadati MEH, Nematian J (2013) Two-level linear programming for fuzzy random portfolio optimization through possibility and necessity-based model. Proc Econ Finance 5:657–666
27. Tanaka H, Guo P (1999) Portfolio selection based on upper and lower exponential possibility distributions. Eur J Oper Res 114:115–126
28. Tanaka H, Guo P, Turksen IB (2000) Portfolio selection based on fuzzy probabilities and possibility distributions. Fuzzy Sets Syst 111:387–397
29. Tasche D (2002) Expected shortfall and beyond. J Bank Finance 26:1519–1533
30. Wang B, Wang S, Watada J (2011) Fuzzy portfolio selection models with value-at-risk. IEEE Trans Fuzzy Syst 19:758–769
31. Watada J (2001) Fuzzy portfolio model for decision making in investment. In: Yoshida Y (ed) Dynamical aspects in fuzzy decision making. Physica, Heidelberg, pp 141–162
32. Yoshida Y (2006) Mean values, measurement of fuzziness and variance of fuzzy random variables for fuzzy optimization. In: Proceedings of SCIS and ISIS 2006, Tokyo, pp 2277–2282
33. Yoshida Y (2007) Fuzzy extension of estimations with randomness: the perception-based approach. In: Proceedings of MDAI 2007, LNAI 4529, Springer, pp 295–306
34. Yoshida Y (2008) Perception-based estimations of fuzzy random variables: linearity and convexity. Int J Uncertain Fuzziness Knowl Based Syst 16(suppl):71–87
35. Yoshida Y (2009) An estimation model of value-at-risk portfolio under uncertainty. Fuzzy Sets Syst 160:3250–3262
36. Yoshida Y (2011) Risk analysis of portfolios under uncertainty: minimizing average rates of falling. J Adv Comput Intell Intell Inform 15:56–62
37. Yoshida Y (2013) Ordered weighted averages on intervals and the sub/super-additivity. J Adv Comput Intell Intell Inform 17(4):520–525
38. Yoshida Y (2017) Portfolios optimization with coherent risk measures in fuzzy asset management. In: Proceedings of ISCBI2017, Dubai, pp 100–104
39. Yoshida Y (2018) Coherent risk measures derived from utility functions. In: Proceedings of MDAI 2018, LNAI, vol 11144. Springer **(to appear)**
40. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353